Continuous state branching processes in random environments.

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CIMAT
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Introduction

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More precisely, a CB-process $Y = (Y_t, t \geq 0)$ is a Markov process taking values in $[0, \infty]$, where 0 and $\infty$ are two absorbing states, and satisfying the branching property.
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More precisely, a CB-process \(Y = (Y_t, t \geq 0)\) is a Markov process taking values in \([0, \infty]\), where 0 and \(\infty\) are two absorbing states, and satisfying the branching property.

In particular,

\[
\mathbb{E}_x \left[ e^{-\lambda Y_t} \right] = \exp\{-xu_t(\lambda)\}, \quad \text{for } \lambda \geq 0,
\]

for some function \(u_t(\lambda)\).
The function $u_t(\lambda)$ is determined by the integral equation

$$
\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(u)} du = t
$$

where $\psi$ (branching mechanism of $Y$) satisfies the Lévy-Khintchine formula

$$
\psi(\lambda) = -a\lambda + \gamma^2 \lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x 1_{\{x < 1\}}) \mu(dx),
$$

where $a \in \mathbb{R}$, $\gamma \geq 0$ and $\mu$ is a $\sigma$-finite measure such that

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\int_{(0,\infty)} (1 \wedge x^2) \mu(dx) < \infty.
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Observe $\mathbb{E}_x[Y_t] = xe^{-\psi'(0^+)t}$. Hence, in respective order, a CB-process is called supercritical, critical or subcritical accordingly as $\psi'(0^+) < 0$, $\psi'(0^+) = 0$ or $\psi'(0^+) > 0$. 
The probability of extinction is given by

\[ P_x \left( \lim_{t \to \infty} Y_t = 0 \right) = e^{-\eta x}, \]

where \( \eta \) is the largest root of \( \psi \).
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A CB-process $Y$ with branching mechanism $\psi$ has a finite time extinction almost surely if and only if

$$
\int_{0}^{\infty} \frac{du}{\psi(u)} < \infty \quad \text{and} \quad \psi'(0+) \geq 0.
$$
A CB-process can also be defined as the unique non-negative strong solution of the stochastic differential equation

\[
Y_t = Y_0 + a \int_0^t Y_s \, ds + \int_0^t \sqrt{2\gamma^2 Y_s} \, dB_s \\
+ \int_0^t \int_{(0,1)} \int_0^{Y_s-} z \tilde{N}(ds, dz, du) + \int_0^t \int_{[1,\infty)} \int_0^{Y_s-} zN(ds, dz, du),
\]

where \( B = (B_t, t \geq 0) \) is a standard Brownian motion, \( N \) is a Poisson random measure independent of \( B \), with intensity \( ds \otimes \mu(dz) \otimes du \) and \( \tilde{N} \) is its compensated version.
CB-process in a Lévy random environment

We introduce a continuous state branching process in a Lévy random environment (CBLRE) as the unique non-negative strong solution of the stochastic differential equation

\[
Z_t = Z_0 + a \int_0^t Z_s \, ds + \int_0^t \sqrt{2 \gamma^2 Z_s} \, dB_s + \int_0^t \int_{(0,1)} \int_0^{Z_s} \tilde{N}(ds, dz, du) + \int_0^t \int_{[1,\infty)} \int_0^{Z_s} zN(ds, dz, du),
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\[ + \int_0^t \int_0^{Z_s} \int_0^{\sigma \tilde{N}(ds, dz, du)} + \int_0^t \int_{[1, \infty)} \int_0^{Z_s} zN(ds, dz, du), \]

where

\[ S_t = \alpha t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} (e^z - 1) \tilde{N}^{(e)}(ds, dz) \]

\[ + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} (e^z - 1) N^{(e)}(ds, dz), \]
with $\alpha \in \mathbb{R}$ and $\sigma \geq 0$, $B^{(e)} = (B^{(e)}_t, t \geq 0)$ is a standard Brownian motion and $N^{(e)}(ds, dz)$ is a Poisson random measure in $\mathbb{R}_+ \times \mathbb{R}$ independent of $B^{(e)}$ with intensity $ds\pi(dy)$, $\tilde{N}^{(e)}$ its compensated version and $\pi$ is a $\sigma$-finite measure satisfying

$$\int_{\mathbb{R}} (1 \wedge z^2)\pi(dz) < \infty.$$ 

We will assume that all the objects involve in the branching and environmental terms are mutually independent.
with $\alpha \in \mathbb{R}$ and $\sigma \geq 0$, $B^{(e)} = (B_t^{(e)}, t \geq 0)$ is a standard Brownian motion and $N^{(e)}(ds, dz)$ is a Poisson random measure in $\mathbb{R}_+ \times \mathbb{R}$ independent of $B^{(e)}$ with intensity $ds \pi(dy)$, $\tilde{N}^{(e)}$ its compensated version and $\pi$ is a $\sigma$-finite measure satisfying

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We define the auxiliary process

$$K_t = mt + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} v\tilde{N}^{(e)}(ds, dv) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} vN^{(e)}(ds, dv),$$

where

$$m = \alpha - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^{uv} - 1 - v) \pi(dv).$$
Let $C^2(\mathbb{R}_+)$ and $D(\mathbb{R}_+)$ be the sets of functions with continues first and second derivatives and the set of càdlàg functions, respectively.
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**Theorem**

*The previous stochastic differential equation has a unique non-negative strong solution. The process $Z = (Z_t, t \geq 0)$ is a Markov process and, conditionally on $K$, it satisfies the branching property.*
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**Theorem**

The previous stochastic differential equation has a unique non-negative strong solution. The process $Z = (Z_t, t \geq 0)$ is a Markov process and, conditionally on $K$, it satisfies the branching property.

Moreover if $|\psi'(0+)| < \infty$, then the auxiliary process can be taken as $K_t^{(0)} = K_t + \psi'(0+)t$ and for any $t > 0$

$$
\mathbb{E}_Z \left[ \exp \left\{ -\lambda Z_t e^{-K_t^{(0)}} \right\} \bigg| K \right] = \exp \left\{ -z v_t(0, \lambda, K^{(0)}) \right\} \quad a.s.,
$$

where for every $(\lambda, \delta) \in (\mathbb{R}_+, D(\mathbb{R}_+))$, $v_t : s \in [0, t] \mapsto v_t(s, \lambda, \delta)$ is the unique solution of the backward differential equation

$$
\frac{\partial}{\partial s} v_t(s, \lambda, K^{(0)}) = e^{K_s^{(0)}} \psi_0(v_t(s, \lambda, K^{(0)}) e^{-K_s^{(0)}}), \quad v_t(t, \lambda, K^{(0)}) = \lambda,
$$
and

\[ \psi_0(\lambda) = \psi(\lambda) - \lambda \psi'(0) = \gamma^2 \lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x) \mu(dx). \]
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**Q: Can we find a unique solution** \( v_t(s, \lambda, K) \) **when** \( \psi(0+) = -\infty \)?
and

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$$

**OQ:** Can we find a unique solution $v_t(s, \lambda, K)$ when $\psi(0+) = -\infty$?

**OQ:** Can we define the SDE of above by replacing the Lévy environment by another process?
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**Neveu’s branching process:** This example correspond to the case when

\[ \psi(u) = u \log u, \quad u \geq 0. \]
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Neveu’s branching process: This example correspond to the case when

\[ \psi(u) = u \log u, \quad u \geq 0. \]

Observe that \( \psi'(0+) = -\infty \). In this case

\[ v_t(s, \lambda, K) = \exp \left\{ e^s \int_s^t e^{-u} K_u \, du + \log \lambda e^{-(t-s)} \right\}. \]
Then,
\[
\mathbb{E}_z \left[ \exp \left\{ -\lambda Z_t e^{-K_t} \right\} \middle| K \right] = \exp \left\{ -z \lambda e^{-t} \exp \left\{ \int_0^t e^{-s} K_s ds \right\} \right\} \quad a.s.,
\]
which implies that
\[
\mathbb{P}_z \left( Z_t > 0 \middle| K \right) = 1, \quad t > 0.
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**Feller’s diffusion** If \( a = \mu(0, \infty) = 0 \), the CBBRE is given by
\[
Z_t = Z_0 + \alpha \int_0^t Z_s ds + \sigma \int_0^t Z_s dS_s + \int_0^t \sqrt{2\gamma^2 Z_s} dB_s.
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\]

**Stable case.** Here, the branching mechanism is of the form
\[
\psi(\lambda) = -a \lambda + c_\beta \lambda^{\beta+1}, \quad \lambda \geq 0,
\]
for some \( \beta \in (-1, 0) \cup (0, 1) \), \( a \in \mathbb{R} \), and
\[
\left\{ \begin{array}{ll}
  c_\beta < 0 & \text{if } \beta \in (-1, 0), \\
  c_\beta > 0 & \text{if } \beta \in (0, 1).
\end{array} \right.
\]
In this case, we note

\[
\psi'(0+) = \begin{cases} 
-\infty & \text{if } \beta \in (-1, 0), \\
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\end{cases} \]

We use in both cases the backward differential equation of Theorem 1 and observe that it satisfies

\[ \frac{\partial}{\partial s} v_t(s, \lambda, \delta) = -av_t(s, \lambda, \delta) + c_\beta v_t^{\beta+1}(s, \lambda, \delta) e^{-\beta \delta s}. \]
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Therefore,

\[ v_t(s, \lambda, \delta) = e^{as} \left( (\lambda e^{at})^{-\beta} + \beta c\beta \int_s^t e^{-\beta(\delta u + au)} du \right)^{-1/\beta}. \]

Implying the following a.s. identity

\[ \mathbb{E}_z \left[ \exp \left\{ -\lambda Z_t e^{-K_t^{(0)}} \right\} \middle| K^{(0)} \right] = \exp \left\{ -z \left( \lambda^{-\beta} + \beta c\beta \int_0^t e^{-\beta K_u^{(0)}} du \right)^{-1/\beta} \right\}. \]
Long-term behaviour

Similarly to the case of CB-processes, there are three events which are of immediate concern for the process $Z$, explosion, absorption and extinction.
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Similarly to the case of CB-processes, there are three events which are of immediate concern for the process $Z$, explosion, absorption and extinction.

Proposition

Assume $|\psi'(0+)| < \infty$, then a CBPBRE $Z$ with branching mechanism $\psi$ satisfies

$$\mathbb{P}_z(Z_t < \infty) = 1, \quad \text{for all } t > 0.$$
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Similarly to the case of CB-processes, there are three events which are of immediate concern for the process $Z$, *explosion*, *absorption* and *extinction*.

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**OQ:** Can we get a necessary and sufficient condition?
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OQ: Can we get a necessary and sufficient condition?

Stable case with $\beta \in (-1, 0)$. From the Laplace transform of $\tilde{Z}$ (taking $\lambda$ goes to 0), we deduce

$$P_z\left( Z_t < \infty \mid K \right) = \exp \left\{ -z \left( \beta c \beta \int_0^t e^{-\beta (K_u + au)} \, du \right)^{-1/\beta} \right\} \quad \text{a.s.,}$$
implying

\[ P_z \left( Z_t = \infty \mid K \right) = 1 - \exp \left\{ -z \left( \beta c_\beta \int_0^t e^{-\beta(K_u + au)} \, du \right)^{-1/\beta} \right\} > 0. \]
implying

\[ \mathbb{P}_z (Z_t = \infty \mid K) = 1 - \exp \left\{ -z \left( \beta c_\beta \int_0^t e^{-\beta(K_u + au)} du \right)^{-1/\beta} \right\} > 0. \]

Moreover, if the process \((K_u + au, u \geq 0)\) does not drift to \(+\infty\), we deduce that \(\lim_{t \to \infty} Z_t = \infty\), a.s.
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$$P_z(Z_t = \infty | K) = 1 - \exp \left\{-z \left( \beta c_\beta \int_0^t e^{-\beta(K_u + au)} du \right)^{-1/\beta} \right\} > 0.$$  

Moreover, if the process \((K_u + au, u \geq 0)\) does not drift to \(+\infty\), we deduce that \(\lim_{t \to \infty} Z_t = \infty\), a.s.

On the other hand, if the process \((K_u + au, u \geq 0)\) drifts to \(+\infty\), we have an interesting behaviour of the process \(Z\),

$$P_z(Z_\infty = \infty) = 1 - \mathbb{E} \left[ \exp \left\{-z \left( \beta c_\beta \int_0^\infty e^{-\beta(K_u + au)} du \right)^{-1/\beta} \right\} \right].$$
implying

\[ \mathbb{P}_z \left( Z_t = \infty \mid K \right) = 1 - \exp \left\{ -z \left( \beta c_\beta \int_0^t e^{-\beta(K_u + au)} \, du \right)^{-1/\beta} \right\} > 0. \]

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**Neveu case.** By taking \(\lambda\) goes to 0 in the Laplace exponent of \(\tilde{Z}\), one can see that the process is conservative conditionally on the environment, i.e.

\[ \mathbb{P}_z (Z_t < \infty \mid K) = 1, \]

for all \(t \in (0, \infty)\) and \(z \in [0, \infty)\).
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i) If $K^{(0)}$ drifts to $-\infty$, then $P_z \left( \lim_{t \to \infty} Z_t = 0 \bigg| K^{(0)} \right) = 1$, a.s.
Proposition

Assume that $|\psi'(0+)| < \infty$. Let $(Z_t, t \geq 0)$ be a CBPBR with branching mechanism given by $\psi$.

i) If $K^{(0)}$ drifts to $-\infty$, then
   \[ P_z \left( \lim_{t \to \infty} Z_t = 0 \mid K^{(0)} \right) = 1, \text{ a.s.} \]

ii) If $K^{(0)}$ oscillates, then
   \[ P_z \left( \liminf_{t \to \infty} Z_t = 0 \mid K^{(0)} \right) = 1, \text{ a.s.} \]

Moreover if $\gamma > 0$ then
   \[ P_z \left( \lim_{t \to \infty} Z_t = 0 \mid K^{(0)} \right) = 1, \text{ a.s.} \]
Proposition

iii) If $K^{(0)}$ drifts to $+\infty$ and

$$\int_{0}^{\infty} x \ln x \mu(dx) < \infty,$$

then $\mathbb{P}_z \left( \lim_{t \to \infty} \inf Z_t > 0 \mid K^{(0)} \right) > 0$ a.s., and there exists a non-negative finite r.v. $W$ such that

$$Z_t e^{-K^{(0)}_t} \xrightarrow{t \to \infty} W, \ a.s \quad \text{and} \quad \{ W = 0 \} = \left\{ \lim_{t \to \infty} Z_t = 0 \right\}.$$

Moreover, if $\gamma > 0$, we have

$$\mathbb{P}_z \left( \lim_{t \to \infty} Z_t = 0 \right) \geq \left( 1 + \frac{z\sigma^2}{\gamma^2} \right)^{-\frac{2m}{\sigma^2}}.$$
OQ: What happen when the integral condition is not satisfied?

It is important to note that in the Feller and stable cases, one can deduce directly that \( \lim_{t \to \infty} Z_t = 0 \), a.s., whenever \( K(0) \) does not drift to \( +\infty \).

In the case when \( K(0) \) drifts to \( +\infty \), we have
\[
P_{Z}(\lim_{t \to \infty} Z_t = 0 | K(0)) = \exp\left\{ -Z(\beta c \int_0^\infty e^{-\beta K(0) u} d\mu) - 1/\beta \right\},
\]
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In the case when $K^{(0)}$ drifts to $+\infty$, we have

$$\mathbb{P}_z \left( \lim_{t \to \infty} Z_t = 0 \mid K^{(0)} \right) = \exp \left\{ -z \left( \beta c_\beta \int_0^\infty e^{-\beta K_u^{(0)}} \, du \right)^{-1/\beta} \right\}, \quad \text{a.s.}$$
Stable case

**Theorem**

Let \((Z_t, t \geq 0)\) be the stable CBLRE with index \(\beta \in (-1, 0)\) and \(Z_0 = z > 0\).
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i) Subcritical-explosion. If \(\phi'_K(0+) < 0\), then there exist \(c_1(z) > 0\) such that
\[
\lim_{t \to \infty} \mathbb{P}_z(Z_t < \infty) = c_1(z).
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ii) Critical-explosion. If \(\phi'_K(0+) < 0\) (+ some moments conditions), then there exist \(c_2(z) > 0\) such that

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\lim_{t \to \infty} \sqrt{t} \mathbb{P}_z(Z_t < \infty) = c_2(z).
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**Theorem**

Let \((Z_t, t \geq 0)\) be the stable CBLRE with index \(\beta \in (-1, 0)\) and \(Z_0 = z > 0\).

i) *Subcritical-explosion.* If \(\phi'_K(0+) < 0\), then there exist \(c_1(z) > 0\) such that

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ii) *Critical-explosion.* If \(\phi'_K(0+) < 0\) (*+ some moments conditions*), then there exist \(c_2(z) > 0\) such that

\[
\lim_{t \to \infty} \sqrt{t} \mathbb{P}_z(Z_t < \infty) = c_2(z).
\]

iii) *Supercritical-explosion.* If \(\phi'_K(0+) < 0\) (*+ some moments conditions*) then there exist \(c_3(z) > 0\)

\[
\lim_{t \to \infty} t^{\frac{3}{2}} e^{\phi_K(\tau)} \mathbb{P}_z(Z_t < \infty) = c_3(z),
\]

where \(\tau\) is the value at which \(\phi_K\) attains its minimum.
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ii) *(Critical case)* If \(\phi'_K(0+) = 0\) *(+ some moments conditions)*, then there exist \(c_5(z) > 0\) such that

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\lim_{t \to \infty} \sqrt{t} \mathbb{P}_z(Z_t > 0) = c_5(z).
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iii) \textit{(Weakly subcritical)} If $\phi'_K(0+) = 0$ and $\phi'_K(1) > 0$ (+ some moments conditions), then there exist $c_6(z) > 0$ such that

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Continuous state branching processes in random environments.

Stable case

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**iv)** (Intermediately subcritical) If $\phi'_K(0+) = 0$ and $\phi'_K(1) = 0$ (+ some moments conditions), then there exist $c_7 > 0$ such that

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**Theorem**

iii) *(Weakly subcritical)* If $\phi_K'(0+) = 0$ and $\phi_K'(1) > 0$ (+ some moments conditions), then there exist $c_6(z) > 0$ such that

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v) *(Strongly subcritical)* If $\phi_K'(0+) = 0$ and $\phi_K'(1) < 0$ (+ some moments conditions), then there exist $c_7 > 0$ such that

$$\lim_{t \to \infty} e^{t\phi_K(1)} P_z(Z_t > 0) = zc_8.$$
More open questions

- Can we go further than the stable case?
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We can construct superprocesses (see Mytnik 96), can we study the event of extinction or local extinction?

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