Continuous state branching processes in random environments.

J.C. Pardo

CIMAT

CB-processes.

A continuous-state branching process (or CB-process) is a non-negative valued strong Markov process with probabilities $(\mathbb{P}_x,\ x \geq 0)$ such that for any $x, y \geq 0$, $\mathbb{P}_{x+y}$ is equal in law to the convolution of $\mathbb{P}_x$ and $\mathbb{P}_y$.

CB-processes.

A continuous-state branching process (or CB-process) is a non-negative valued strong Markov process with probabilities $(\mathbb{P}_x,\, x \geq 0)$ such that for any $x, y \geq 0$, $\mathbb{P}_{x+y}$ is equal in law to the convolution of $\mathbb{P}_x$ and $\mathbb{P}_y$.

CB-processes may be thought of as the continuous (in time and space) analogues of classical Galton-Watson branching processes.

CB-processes.

A continuous-state branching process (or CB-process) is a non-negative valued strong Markov process with probabilities $(\mathbb{P}_x,\ x \geq 0)$ such that for any $x, y \geq 0$, $\mathbb{P}_{x+y}$ is equal in law to the convolution of $\mathbb{P}_x$ and $\mathbb{P}_y$.

CB-processes may be thought of as the continuous (in time and space) analogues of classical Galton-Watson branching processes.

More precisely, a CB-process $Y = (Y_t, t \geq 0)$ is a Markov process taking values in $[0, \infty]$, where 0 and ∞ are two absorbing states, and satisfying the branching property.

CB-processes.

A continuous-state branching process (or CB-process) is a non-negative valued strong Markov process with probabilities $(\mathbb{P}_x,\ x \geq 0)$ such that for any $x, y \geq 0$, $\mathbb{P}_{x+y}$ is equal in law to the convolution of $\mathbb{P}_x$ and $\mathbb{P}_y$.

CB-processes may be thought of as the continuous (in time and space) analogues of classical Galton-Watson branching processes.

More precisely, a CB-process $Y = (Y_t, t \geq 0)$ is a Markov process taking values in $[0, \infty]$, where 0 and ∞ are two absorbing states, and satisfying the branching property.

In particular,

$$\mathbb{E}_x\left[e^{-\lambda Y_t}\right] = \exp\{-xu_t(\lambda)\}, \qquad \text{for } \lambda \geq 0,$$

for some function $u_t(\lambda)$.

The function $u_t(\lambda)$ is determined by the integral equation

$$\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(u)} \mathrm{d}u = t$$

where ψ (branching mechanism of Y) satisfies the Lévy-Khincthine formula

$$\psi(\lambda) = -a\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}}\right)\mu(\mathrm{d}x),$$

where $a \in \mathbb{R}$, $\gamma \geq 0$ and μ is a σ-finite measure such that

$$\int_{(0,\infty)} (1 \wedge x^2)\mu(\mathrm{d}x) < \infty.$$

The function $u_t(\lambda)$ is determined by the integral equation

$$\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(u)} \mathrm{d}u = t$$

where ψ (branching mechanism of Y) satisfies the Lévy-Khincthine formula

$$\psi(\lambda) = -a\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}}\right)\mu(\mathrm{d}x),$$

where $a \in \mathbb{R}$, $\gamma \geq 0$ and μ is a σ-finite measure such that

$$\int_{(0,\infty)} (1 \wedge x^2)\mu(\mathrm{d}x) < \infty.$$

Observe $\mathbb{E}_x[Y_t] = xe^{-\psi'(0^+)t}$. Hence, in respective order, a CB-process is called supercritical, critical or subcritical accordingly as $\psi'(0^+) < 0$, $\psi'(0^+) = 0$ or $\psi'(0^+) > 0$.

The probability of extinction is given by

$$\mathbb{P}_x\left(\lim_{t\to\infty} Y_t = 0\right) = e^{-\eta x},$$

where η is the largest root of ψ.

The probability of extinction is given by

$$\mathbb{P}_x\left(\lim_{t\to\infty} Y_t = 0\right) = e^{-\eta x},$$

where η is the largest root of ψ.

A CB-process Y with branching mechanism ψ has a finite time extinction almost surely if and only if

$$\int^{\infty} \frac{\mathrm{d}u}{\psi(u)} < \infty \quad \text{and} \quad \psi'(0+) \geq 0.$$

A CB-process can also be defined as the unique non-negative strong solution of the stochastic differential equation

$$Y_t = Y_0 + a\int_0^t Y_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s + \int_0^t \int_{(0,1)} \int_0^{Y_{s-}} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{Y_{s-}} zN(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

where $B = (B_t, t \geq 0)$ is a standard Brownian motion, N is a Poisson random measure independent of B, with intensity $\mathrm{d}s \otimes \mu(\mathrm{d}z) \otimes \mathrm{d}u$ and $\tilde{N}$ is its compensated version.

CB-process in a Lévy random environment

We introduce a continuous state branching process in a Lévy random environment (CBLRE) as the unique non-negative strong solution of the stochastic differential equation

$$Z_t = Z_0 + a\int_0^t Z_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s + \int_0^t \int_{(0,1)} \int_0^{Z_{s-}} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{Z_{s-}} zN(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

CB-process in a Lévy random environment

We introduce a continuous state branching process in a Lévy random environment (CBLRE) as the unique non-negative strong solution of the stochastic differential equation

$$\begin{aligned} Z_t =& Z_0 + a\int_0^t Z_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s + \int_0^t Z_s \mathrm{d}S_s \\ &+ \int_0^t \int_{(0,1)} \int_0^{Z_{s-}} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{Z_{s-}} zN(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned}$$

where

$$\begin{aligned} S_t = \alpha t + \sigma B_t^{(e)} &+ \int_0^t \int_{(-1,1)} (e^z - 1)\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z) \\ &+ \int_0^t \int_{\mathbb{R}\setminus(-1,1)} (e^z - 1)N^{(e)}(\mathrm{d}s, \mathrm{d}z), \end{aligned}$$

with $\alpha \in \mathbb{R}$ and $\sigma \geq 0$, $B^{(e)} = (B_t^{(e)}, t \geq 0)$ is a standard Brownian motion and $N^{(e)}(\mathrm{d}s, \mathrm{d}z)$ is a Poisson random measure in $\mathbb{R}_+ \times \mathbb{R}$ independent of $B^{(e)}$ with intensity $\mathrm{d}s\pi(\mathrm{d}y)$, $\widetilde{N}^{(e)}$ its compensated version and π is a σ-finite measure satisfying

$$\int_{\mathbb{R}} (1 \wedge z^2)\pi(\mathrm{d}z) < \infty.$$

We will assume that all the objects involve in the branching and environmental terms are mutually independent.

with $\alpha \in \mathbb{R}$ and $\sigma \geq 0$, $B^{(e)} = (B_t^{(e)}, t \geq 0)$ is a standard Brownian motion and $N^{(e)}(\mathrm{d}s, \mathrm{d}z)$ is a Poisson random measure in $\mathbb{R}_+ \times \mathbb{R}$ independent of $B^{(e)}$ with intensity $\mathrm{d}s\pi(\mathrm{d}y)$, $\widetilde{N}^{(e)}$ its compensated version and π is a σ-finite measure satisfying

$$\int_{\mathbb{R}} (1 \wedge z^2)\pi(\mathrm{d}z) < \infty.$$

We will assume that all the objects involve in the branching and environmental terms are mutually independent.

We define the auxiliary process

$$K_t = \mathbf{m}t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} v\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}v) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} vN^{(e)}(\mathrm{d}s, \mathrm{d}v),$$

where

$$\mathbf{m} = \alpha - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^v - 1 - v)\pi(\mathrm{d}v).$$

Let $C^2(\mathbb{R}_+)$ and $D(\mathbb{R}_+)$ be the sets of functions with continues first and second derivatives and the set of càdlàg functions, respectively.

Let $C^2(\mathbb{R}_+)$ and $D(\mathbb{R}_+)$ be the sets of functions with continues first and second derivatives and the set of càdlàg functions, respectively.

Theorem

The previous stochastic differential equation has a unique non-negative strong solution. The process $Z = (Z_t, t \geq 0)$ is a Markov process and, conditionally on K, it satisfies the branching property.

Let $C^2(\mathbb{R}_+)$ and $D(\mathbb{R}_+)$ be the sets of functions with continues first and second derivatives and the set of càdlàg functions, respectively.

Theorem

The previous stochastic differential equation has a unique non-negative strong solution. The process $Z = (Z_t, t \geq 0)$ is a Markov process and, conditionally on K, it satisfies the branching property.

Moreover if $|\psi'(0+)| < \infty$, then the auxiliary process can be taken as $K_t^{(0)} = K_t + \psi'(0+)t$ and for any $t > 0$

$$\mathbb{E}_z\Big[\exp\Big\{ -\lambda Z_t e^{-K_t^{(0)}}\Big\}\Big|K\Big] = \exp\Big\{ -z v_t(0,\lambda,K^{(0)})\Big\} \qquad a.s.,$$

where for every $(\lambda,\delta) \in (\mathbb{R}_+, D(\mathbb{R}_+))$, $v_t : s \in [0,t] \mapsto v_t(s,\lambda,\delta)$ is the unique solution of the backward differential equation

$$\frac{\partial}{\partial s} v_t(s,\lambda,K^{(0)}) = e^{K_s^{(0)}}\psi_0(v_t(s,\lambda,K^{(0)})e^{-K_s^{(0)}}), \qquad v_t(t,\lambda,K^{(0)}) = \lambda,$$

and

$$\psi_0(\lambda) = \psi(\lambda) - \lambda\psi'(0) = \gamma^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x\right)\mu(\mathrm{d}\, x).$$

and

$$\psi_0(\lambda) = \psi(\lambda) - \lambda\psi'(0) = \gamma^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x\right)\mu(\mathrm{d}\, x).$$

OQ: Can we find a unique solution $v_t(s, \lambda, K)$ when $\psi(0+) = -\infty$?

and

$$\psi_0(\lambda) = \psi(\lambda) - \lambda\psi'(0) = \gamma^2\lambda^2 + \int_{(0,\infty)} \big(e^{-\lambda x} - 1 + \lambda x\big)\mu(\mathrm{d}\, x).$$

OQ: Can we find a unique solution $v_t(s, \lambda, K)$ when $\psi(0+) = -\infty$?

OQ: Can we define the SDE of above by replacing the Lévy environment by another process?

and

$$\psi_0(\lambda) = \psi(\lambda) - \lambda\psi'(0) = \gamma^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x\right)\mu(\mathrm{d}\, x).$$

OQ: Can we find a unique solution $v_t(s, \lambda, K)$ when $\psi(0+) = -\infty$?

OQ: Can we define the SDE of above by replacing the Lévy environment by another process? Actually, we can replace the Lévy environment by a Markov additive environment (Palau & P.)

and

$$\psi_0(\lambda) = \psi(\lambda) - \lambda\psi'(0) = \gamma^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x\right)\mu(\mathrm{d}\,x).$$

OQ: Can we find a unique solution $v_t(s,\lambda,K)$ when $\psi(0+) = -\infty$?

OQ: Can we define the SDE of above by replacing the Lévy environment by another process? Actually, we can replace the Lévy environment by a Markov additive environment (Palau & P.)

Neveu's branching process: This example correspond to the case when

$$\psi(u) = u\log u, \qquad u \geq 0.$$

and

$$\psi_0(\lambda) = \psi(\lambda) - \lambda\psi'(0) = \gamma^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x\right)\mu(\mathrm{d}\,x).$$

OQ: Can we find a unique solution $v_t(s, \lambda, K)$ when $\psi(0+) = -\infty$?

OQ: Can we define the SDE of above by replacing the Lévy environment by another process? Actually, we can replace the Lévy environment by a Markov additive environment (Palau & P.)

Neveu's branching process: This example correspond to the case when

$$\psi(u) = u \log u, \qquad u \geq 0.$$

Observe that $\psi'(0+) = -\infty$. In this case

$$v_t(s, \lambda, K) = \exp\left\{ e^s \int_s^t e^{-u} K_u \mathrm{d}u + \log \lambda e^{-(t-s)} \right\}.$$

Then,

$$\mathbb{E}_z\left[\exp\left\{-\lambda Z_t e^{-K_t}\right\}\Big| K\right] = \exp\left\{-z\lambda^{e^{-t}} \exp\left\{\int_0^t e^{-s} K_s \mathrm{d}s\right\}\right\} \qquad a.s.,$$

which implies that

$$\mathbb{P}_z\left(Z_t > 0 \Big| K\right) = 1, \qquad t > 0.$$

Then,

$$\mathbb{E}_z\left[\exp\left\{-\lambda Z_t e^{-K_t}\right\}\Big|K\right] = \exp\left\{-z\lambda^{e^{-t}} \exp\left\{\int_0^t e^{-s} K_s \mathrm{d}s\right\}\right\} \qquad a.s.,$$

which implies that

$$\mathbb{P}_z\left(Z_t > 0\Big|K\right) = 1, \qquad t > 0.$$

Feller's diffusion If $a = \mu(0,\infty) = 0$, the CBBRE is given by

$$Z_t = Z_0 + \alpha\int_0^t Z_s\,ds + \sigma\int_0^t Z_s\,dS_s + \int_0^t \sqrt{2\gamma^2 Z_s}\,dB_s.$$

Then,

$$\mathbb{E}_z\left[\exp\left\{-\lambda Z_t e^{-K_t}\right\}\Big|K\right] = \exp\left\{-z\lambda^{e^{-t}}\exp\left\{\int_0^t e^{-s}K_s \mathrm{d}s\right\}\right\} \qquad a.s.,$$

which implies that

$$\mathbb{P}_z\left(Z_t > 0\Big|K\right) = 1, \qquad t > 0.$$

Feller's diffusion If $a = \mu(0,\infty) = 0$, the CBBRE is given by

$$Z_t = Z_0 + \alpha\int_0^t Z_s\,ds + \sigma\int_0^t Z_s\,dS_s + \int_0^t \sqrt{2\gamma^2 Z_s}\,dB_s.$$

Stable case. Here, the branching mechanism is of the form

$$\psi(\lambda) = -a\lambda + c_\beta\lambda^{\beta+1}, \qquad \lambda \geq 0,$$

for some $\beta \in (-1,0)\cup(0,1)$, $a \in \mathbb{R}$, and

$$\begin{cases} c_\beta < 0 & \text{if } \beta \in (-1,0), \\ c_\beta > 0 & \text{if } \beta \in (0,1). \end{cases}$$

In this case, we note

$$\psi'(0+) = \begin{cases} -\infty & \text{if } \beta \in (-1, 0), \\ -a & \text{if } \beta \in (0, 1). \end{cases}$$

In this case, we note

$$\psi'(0+) = \begin{cases} -\infty & \text{if } \beta \in (-1, 0), \\ -a & \text{if } \beta \in (0, 1). \end{cases}$$

We use in both cases the backward differential equation of Theorem 1 and observe that it satisfies

$$\frac{\partial}{\partial s} v_t(s, \lambda, \delta) = -a v_t(s, \lambda, \delta) + c_\beta v_t^{\beta+1}(s, \lambda, \delta) e^{-\beta \delta_s}.$$

In this case, we note

$$\psi'(0+) = \begin{cases} -\infty & \text{if } \beta \in (-1,0), \\ -a & \text{if } \beta \in (0,1). \end{cases}$$

We use in both cases the backward differential equation of Theorem 1 and observe that it satisfies

$$\frac{\partial}{\partial s} v_t(s,\lambda,\delta) = -a v_t(s,\lambda,\delta) + c_\beta v_t^{\beta+1}(s,\lambda,\delta) e^{-\beta\delta_s}.$$

Therefore,

$$v_t(s,\lambda,\delta) = e^{as}\left((\lambda e^{at})^{-\beta} + \beta c_\beta \int_s^t e^{-\beta(\delta_u + au)} \mathrm{d}u\right)^{-1/\beta}.$$

Implying the following a.s. identity

$$\mathbb{E}_z\left[\exp\left\{-\lambda Z_t e^{-K_t^{(0)}}\right\} \Big| K^{(0)}\right] = \exp\left\{-z\left(\lambda^{-\beta} + \beta c_\beta \int_0^t e^{-\beta K_u^{(0)}} \mathrm{d}u\right)^{-1/\beta}\right\}.$$

Long-term behaviour

Similarly to the case of CB-processes, there are three events which are of immediate concern for the process Z, *explosion*, *absorption* and *extinction*.

Long-term behaviour

Similarly to the case of CB-processes, there are three events which are of immediate concern for the process Z, *explosion*, *absorption* and *extinction*.

Proposition

Assume $|\psi'(0+)| < \infty$, *then a CBPBRE* Z *with branching mechanism* ψ *satisfies*

$$\mathbb{P}_z(Z_t < \infty) = 1, \qquad \textit{for all} \ \ t > 0.$$

Long-term behaviour

Similarly to the case of CB-processes, there are three events which are of immediate concern for the process Z, *explosion*, *absorption* and *extinction*.

Proposition

Assume $|\psi'(0+)| < \infty$, then a CBPBRE Z with branching mechanism ψ satisfies

$$\mathbb{P}_z(Z_t < \infty) = 1, \qquad \textit{for all} \quad t > 0.$$

OQ: Can we get a necessary and sufficient condition?

Long-term behaviour

Similarly to the case of CB-processes, there are three events which are of immediate concern for the process Z, *explosion*, *absorption* and *extinction*.

Proposition

Assume $|\psi'(0+)| < \infty$, *then a CBPBRE* Z *with branching mechanism* ψ *satisfies*

$$\mathbb{P}_z(Z_t < \infty) = 1, \qquad \textit{for all} \quad t > 0.$$

OQ: Can we get a necessary and sufficient condition?

Stable case with $\beta \in (-1, 0)$. From the Laplace transform of $\tilde{Z}$ (taking λ goes to 0), we deduce

$$\mathbb{P}_z\Big(Z_t < \infty \Big| K\Big) = \exp\left\{-z\left(\beta c_\beta \int_0^t e^{-\beta(K_u + au)} \mathrm{d}u\right)^{-1/\beta}\right\} \qquad \text{a.s.},$$

implying

$$\mathbb{P}_z\left(Z_t=\infty\middle|K\right)=1-\exp\left\{-z\left(\beta c_\beta\int_0^t e^{-\beta(K_u+au)}\mathrm{d}u\right)^{-1/\beta}\right\}>0.$$

implying

$$\mathbb{P}_z\left(Z_t = \infty \middle| K\right) = 1 - \exp\left\{ -z\left(\beta c_\beta \int_0^t e^{-\beta(K_u + au)} \mathrm{d}u\right)^{-1/\beta} \right\} > 0.$$

Moreover, if the process $(K_u + au, u \geq 0)$ does not drift to $+\infty$, we deduce that $\lim_{t\to\infty} Z_t = \infty$, a.s.

implying

$$\mathbb{P}_z\left(Z_t = \infty \middle| K\right) = 1 - \exp\left\{-z\left(\beta c_\beta \int_0^t e^{-\beta(K_u + au)} \mathrm{d}u\right)^{-1/\beta}\right\} > 0.$$

Moreover, if the process $(K_u + au, u \geq 0)$ does not drift to $+\infty$, we deduce that $\lim_{t\to\infty} Z_t = \infty$, a.s.

On the other hand, if the process $(K_u + au, u \geq 0)$ drifts to $+\infty$, we have an interesting behaviour of the process Z,

$$\mathbb{P}_z\left(Z_\infty = \infty\right) = 1 - \mathbb{E}\left[\exp\left\{-z\left(\beta c_\beta \int_0^\infty e^{-\beta(K_u + au)} \mathrm{d}u\right)^{-1/\beta}\right\}\right].$$

implying

$$\mathbb{P}_z\Big(Z_t=\infty\Big|K\Big)=1-\exp\left\{-z\left(\beta c_\beta\int_0^t e^{-\beta(K_u+au)}\mathrm{d}u\right)^{-1/\beta}\right\}>0.$$

Moreover, if the process $(K_u+au, u\geq 0)$ does not drift to $+\infty$, we deduce that $\lim_{t\to\infty} Z_t=\infty$, a.s.

On the other hand, if the process $(K_u+au, u\geq 0)$ drifts to $+\infty$, we have an interesting behaviour of the process Z,

$$\mathbb{P}_z\Big(Z_\infty=\infty\Big)=1-\mathbb{E}\left[\exp\left\{-z\left(\beta c_\beta\int_0^\infty e^{-\beta(K_u+au)}\mathrm{d}u\right)^{-1/\beta}\right\}\right].$$

Neveu case. By taking λ goes to 0 in the Laplace exponent of $\tilde{Z}$, one can see that the process is conservative conditionally on the environment, i.e.

$$\mathbb{P}_z(Z_t<\infty|K)=1,$$

for all $t\in(0,\infty)$ and $z\in[0,\infty)$.

Proposition

Assume that $|\psi'(0+)| < \infty$. *Let* $(Z_t, t \geq 0)$ *be a CBPBRE with branching mechanism given by* ψ.

Proposition

Assume that $|\psi'(0+)| < \infty$. *Let* $(Z_t, t \geq 0)$ *be a CBPBRE with branching mechanism given by* ψ.

i) *If* $K^{(0)}$ *drifts to* $-\infty$, *then* $\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0 \middle| K^{(0)}\right) = 1$, *a.s.*

Proposition

Assume that $|\psi'(0+)| < \infty$. Let $(Z_t, t \geq 0)$ be a CBPBRE with branching mechanism given by ψ.

- **i)** *If $K^{(0)}$ drifts to $-\infty$, then $\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0 \middle| K^{(0)}\right) = 1$, a.s.*
- **ii)** *If $K^{(0)}$ oscillates, then $\mathbb{P}_z\left(\liminf_{t\to\infty} Z_t = 0 \middle| K^{(0)}\right) = 1$, a.s.*

  Moreover if $\gamma > 0$ then

$$\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0 \middle| K^{(0)}\right) = 1, a.s.$$

Proposition

iii) *If* $K^{(0)}$ *drifts to* $+\infty$ *and*

$$\int^{\infty} x \ln x \, \mu(\mathrm{d}x) < \infty,$$

then $\mathbb{P}_z\left(\liminf_{t\to\infty} Z_t > 0 \,\middle|\, K^{(0)}\right) > 0$ *a.s., and there exists a non-negative finite r.v.* W *such that*

$$Z_t e^{-K_t^{(0)}} \underset{t\to\infty}{\longrightarrow} W, \ a.s \qquad and \qquad \{W = 0\} = \left\{\lim_{t\to\infty} Z_t = 0\right\}.$$

Moreover if $\gamma > 0$*, we have*

$$\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0\right) \geq \left(1 + \frac{z\sigma^2}{\gamma^2}\right)^{-\frac{2\mathbf{m}}{\sigma^2}}.$$

OQ: What happen when the integral condition is not satisfied?

OQ: What happen when the integral condition is not satisfied?

It is important to note that in the Feller and stable cases, one can deduce directly that

$$\lim_{t\to\infty} Z_t = 0, \quad \text{a.s.},$$

whenever $K^{(0)}$ does not drift to $+\infty$.

OQ: What happen when the integral condition is not satisfied?

It is important to note that in the Feller and stable cases, one can deduce directly that

$$\lim_{t\to\infty} Z_t = 0, \quad \text{a.s.},$$

whenever $K^{(0)}$ does not drift to $+\infty$.

In the case when $K^{(0)}$ drifts to $+\infty$, we have

$$\mathbb{P}_z(\lim_{t\to\infty} Z_t = 0 | K^{(0)}) = \exp\left\{-z\left(\beta c_\beta \int_0^\infty e^{-\beta K_u^{(0)}} \mathrm{d}u\right)^{-1/\beta}\right\}, \qquad \text{a.s.}$$

Stable case

Theorem

Let $(Z_t, t \geq 0)$ be the stable CBLRE with index $\beta \in (-1, 0)$ and $Z_0 = z > 0$.

Stable case

Theorem

Let $(Z_t, t \geq 0)$ *be the stable CBLRE with index* $\beta \in (-1, 0)$ *and* $Z_0 = z > 0$.

i) *Subcritical-explosion. If* $\phi'_K(0+) < 0$, *then there exist* $c_1(z) > 0$ *such that*

$$\lim_{t \to \infty} \mathbb{P}_z(Z_t < \infty) = c_1(z).$$

Stable case

Theorem

Let $(Z_t, t \geq 0)$ *be the stable CBLRE with index* $\beta \in (-1, 0)$ *and* $Z_0 = z > 0$.

i) *Subcritical-explosion. If* $\phi'_K(0+) < 0$, *then there exist* $c_1(z) > 0$ *such that*

$$\lim_{t \to \infty} \mathbb{P}_z(Z_t < \infty) = c_1(z).$$

ii) *Critical-explosion. If* $\phi'_K(0+) < 0$ *(+ some moments conditions), then then there exist* $c_2(z) > 0$ *such that*

$$\lim_{t \to \infty} \sqrt{t}\mathbb{P}_z(Z_t < \infty) = c_2(z).$$

Stable case

Theorem

Let $(Z_t, t \geq 0)$ *be the stable CBLRE with index* $\beta \in (-1, 0)$ *and* $Z_0 = z > 0$.

i) *Subcritical-explosion. If* $\phi'_K(0+) < 0$, *then there exist* $c_1(z) > 0$ *such that*

$$\lim_{t\to\infty} \mathbb{P}_z(Z_t < \infty) = c_1(z).$$

ii) *Critical-explosion. If* $\phi'_K(0+) < 0$ *(+ some moments conditions), then then there exist* $c_2(z) > 0$ *such that*

$$\lim_{t\to\infty} \sqrt{t}\mathbb{P}_z(Z_t < \infty) = c_2(z).$$

iii) *Supercritical-explosion. If* $\phi'_K(0+) < 0$ *(+ some moments conditions) then there exist* $c_3(z) > 0$

$$\lim_{t\to\infty} t^{\frac{3}{2}} e^{\phi_K(\tau)}\mathbb{P}_z(Z_t < \infty) = c_3(z),$$

where τ *is the value at which* ϕ_K *attains its minimum.*

Theorem

Let $(Z_t, t \geq 0)$ *be a the stable CBLRE with* $\beta \in (0,1)$*. Then for all* $z > 0$,

Theorem

Let $(Z_t, t \geq 0)$ *be a the stable CBLRE with* $\beta \in (0,1)$. *Then for all* $z > 0$,

i) *(Supercritical case) If* $\phi'_K(0+) > 0$ *(+ some moments conditions), then there exist* $c_4(z) > 0$ *such that*

$$\lim_{t\to\infty} \mathbb{P}_z(Z_t > 0) = c_4(z).$$

Theorem

Let $(Z_t, t \geq 0)$ be a the stable CBLRE with $\beta \in (0,1)$. Then for all $z > 0$,

i) *(Supercritical case) If $\phi'_K(0+) > 0$ (+ some moments conditions), then there exist $c_4(z) > 0$ such that*

$$\lim_{t\to\infty} \mathbb{P}_z(Z_t > 0) = c_4(z).$$

ii) *(Critical case) If $\phi'_K(0+) = 0$ (+ some moments conditions), then there exist $c_5(z) > 0$ such that*

$$\lim_{t\to\infty} \sqrt{t}\mathbb{P}_z(Z_t > 0) = c_5(z).$$

Theorem

iii) *(Weakly subcritical) If* $\phi'_K(0+) = 0$ *and* $\phi'_K(1) > 0$ *(+ some moments conditions), then there exist* $c_6(z) > 0$ *such that*

$$\lim_{t\to\infty} t^{\frac{3}{2}} e^{\phi_K(\tau)} \mathbb{P}_z(Z_t > 0) = c_6(z),$$

where τ *is the value at which* ϕ_K *attains its minimum.*

Theorem

iii) *(Weakly subcritical) If $\phi'_K(0+) = 0$ and $\phi'_K(1) > 0$ (+ some moments conditions), then there exist $c_6(z) > 0$ such that*

$$\lim_{t\to\infty} t^{\frac{3}{2}} e^{\phi_K(\tau)} \mathbb{P}_z(Z_t > 0) = c_6(z),$$

where τ is the value at which ϕ_K attains its minimum.

iv) *(Intermediately subcritical) If $\phi'_K(0+) = 0$ and $\phi'_K(1) = 0$ (+ some moments conditions), then there exist $c_7 > 0$ such that*

$$\lim_{t\to\infty} \sqrt{t} e^{\phi_K(1)} \mathbb{P}_z(Z_t > 0) = z c_7.$$

Theorem

iii) *(Weakly subcritical) If* $\phi'_K(0+) = 0$ *and* $\phi'_K(1) > 0$ *(+ some moments conditions), then there exist* $c_6(z) > 0$ *such that*

$$\lim_{t\to\infty} t^{\frac{3}{2}} e^{\phi_K(\tau)} \mathbb{P}_z(Z_t > 0) = c_6(z),$$

where τ *is the value at which* ϕ_K *attains its minimum.*

iv) *(Intermediately subcritical) If* $\phi'_K(0+) = 0$ *and* $\phi'_K(1) = 0$ *(+ some moments conditions), then there exist* $c_7 > 0$ *such that*

$$\lim_{t\to\infty} \sqrt{t} e^{\phi_K(1)} \mathbb{P}_z(Z_t > 0) = zc_7.$$

v) *(Strongly subcritical)If* $\phi'_K(0+) = 0$ *and* $\phi'_K(1) < 0$ *(+ some moments conditions), then there exist* $c_7 > 0$ *such that*

$$\lim_{t\to\infty} e^{t\phi_K(1)} \mathbb{P}_z(Z_t > 0) = zc_8.$$

More open questions

- Can we go further than the stable case ?

More open questions

- Can we go further than the stable case ? Bansaye, P. and Smadi are studying the case when $\psi(\lambda) > \lambda^{1+\epsilon}$, for $\epsilon > 0$ and K satisfies

$$\lim_{t\to\infty} \mathbb{P}(K_t > 0) = \rho \in [0,1].$$

More open questions

- Can we go further than the stable case ? Bansaye, P. and Smadi are studying the case when $\psi(\lambda) > \lambda^{1+\epsilon}$, for $\epsilon > 0$ and K satisfies

$$\lim_{t\to\infty} \mathbb{P}(K_t > 0) = \rho \in [0, 1].$$

- Can we study its genealogy?

More open questions

- Can we go further than the stable case ? Bansaye, P. and Smadi are studying the case when $\psi(\lambda) > \lambda^{1+\epsilon}$, for $\epsilon > 0$ and K satisfies

$$\lim_{t\to\infty} \mathbb{P}(K_t > 0) = \rho \in [0,1].$$

- Can we study its genealogy? Palau is studying how the prolific individuals are affected by the environment.

More open questions

- Can we go further than the stable case ? Bansaye, P. and Smadi are studying the case when $\psi(\lambda) > \lambda^{1+\epsilon}$, for $\epsilon > 0$ and K satisfies

$$\lim_{t\to\infty} \mathbb{P}(K_t > 0) = \rho \in [0,1].$$

- Can we study its genealogy? Palau is studying how the prolific individuals are affected by the environment.
- What about quasi-stationary distributions?

More open questions

- Can we go further than the stable case ? Bansaye, P. and Smadi are studying the case when $\psi(\lambda) > \lambda^{1+\epsilon}$, for $\epsilon > 0$ and K satisfies

$$\lim_{t\to\infty} \mathbb{P}(K_t > 0) = \rho \in [0,1].$$

- Can we study its genealogy? Palau is studying how the prolific individuals are affected by the environment.
- What about quasi-stationary distributions?
- In the stable case, there must be another regime for the process conditioned to die-out.

More open questions

- Can we go further than the stable case ? Bansaye, P. and Smadi are studying the case when $\psi(\lambda) > \lambda^{1+\epsilon}$, for $\epsilon > 0$ and K satisfies

$$\lim_{t\to\infty} \mathbb{P}(K_t > 0) = \rho \in [0,1].$$

- Can we study its genealogy? Palau is studying how the prolific individuals are affected by the environment.
- What about quasi-stationary distributions?
- In the stable case, there must be another regime for the process conditioned to die-out.
- We can construct superprocesses (see Mytnik 96), can we study the event of extinction or local extinction?

More open questions

- Can we go further than the stable case ? Bansaye, P. and Smadi are studying the case when $\psi(\lambda) > \lambda^{1+\epsilon}$, for $\epsilon > 0$ and K satisfies

$$\lim_{t\to\infty} \mathbb{P}(K_t > 0) = \rho \in [0, 1].$$

- Can we study its genealogy? Palau is studying how the prolific individuals are affected by the environment.
- What about quasi-stationary distributions?
- In the stable case, there must be another regime for the process conditioned to die-out.
- We can construct superprocesses (see Mytnik 96), can we study the event of extinction or local extinction?
- What about competition models?

More open questions

- Can we go further than the stable case ? Bansaye, P. and Smadi are studying the case when $\psi(\lambda) > \lambda^{1+\epsilon}$, for $\epsilon > 0$ and K satisfies

$$\lim_{t \to \infty} \mathbb{P}(K_t > 0) = \rho \in [0, 1].$$

- Can we study its genealogy? Palau is studying how the prolific individuals are affected by the environment.
- What about quasi-stationary distributions?
- In the stable case, there must be another regime for the process conditioned to die-out.
- We can construct superprocesses (see Mytnik 96), can we study the event of extinction or local extinction?
- What about competition models? Gonzalez-Casanova, P. and Perez are studying a particular model which is linked to fragmentation-coalescence processes.