Continuous-state branching processes

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Definiton

- A stochastic process $(X_t : t \geq 0)$ with probabilities $(\mathbb{P}_x,\ x \geq 0)$ on $D(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$\mathbb{E}_{x+y}[\mathrm{e}^{-\lambda X_t}] = \mathbb{E}_x[\mathrm{e}^{-\lambda X_t}]\mathbb{E}_y[\mathrm{e}^{-\lambda X_t}], \qquad \lambda \geq 0, t \geq 0.$$

  (written in shorthand $\mathbb{P}_{x+y} = \mathbb{P}_x \otimes \mathbb{P}_y$).
- The transition semigroup is characterised by

$$\mathbb{E}_x[\mathrm{e}^{-\lambda X_t}] = \mathrm{e}^{-u_t(\lambda)x}, \qquad \lambda \geq 0, t \geq 0.$$

  where

$$u_t(\lambda) = \lambda - \int_0^t \psi(u_s(\lambda))\mathrm{d}s, \qquad t \geq 0$$

  such that

$$\psi(\lambda) = -q - a\lambda + \sigma\lambda^2 + \int_{(0,\infty)} (\mathrm{e}^{-\lambda x} - 1 + \lambda \mathbf{1}_{(x<1)} x)\Pi(\mathrm{d}x), \qquad \lambda \geq 0,$$

  with $a \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$.

Conservative, Extinction, Extinguishing and Criticality

- **Conservative:** To avoid the event of explosion $\{\exists \zeta_\infty > 0 : X_t = \infty \ \forall t \geq \zeta_\infty\}$ occurring with positive probability, we have the necessary and sufficient conditions

$$\int_{0+} \frac{1}{|\psi(u)|} \mathrm{d}u = \infty$$

- **Extinction vs Extinguishing:** There are two different ways that a CSBP can 'die out':
  Extinction: $\exists \zeta_0 : X_t = 0 \forall t \geq \zeta_0$
  Extinguishing: $\lim_{t\to\infty} X_t = 0$, $X_t > 0 \ \forall t \geq 0$.
- Extinction if and only if

$$\int^{\infty} \frac{1}{\psi(u)} \mathrm{d}u < \infty.$$

- **Criticality:** Just like Galton-Watson processes there is exponential mean growth:

$$\mathbb{E}_x[X_t] = x\mathrm{e}^{-\psi'(0+)t}$$

  Hence subcritical/supercritical/critical accordingly as $\psi'(0+) > 0/\psi'(0+) < 0/\psi'(0+) = 0$.

Continuous-time Galton–Watson processes and compound Poisson

- Write $\{Z(t) : t \geq 0\}$ for the number of individuals at time t in a continuous-time GW process with offspring distribution p_i, $i \geq 0$.
- Introduce a new distribution on $\{\pi_i : i = -1, 0, 1, 2, \cdots\}$, where $\pi_i = p_{i+1}$. (The number of GW offspring minus 1).
- Write, for $t \geq 0$,

$$J_t = \int_0^t Z(s)\mathrm{d}s, \qquad \varphi(t) = \inf\{u > 0 : J_u > t\}$$

(with the usual $\inf \emptyset = \infty$) and define

$$L(t) = Z(\varphi(t)), \qquad t \geq 0.$$

- Consider what happens up to the first branching time T_1:
- If $Z(0) = k$, then T_1 is the minimum of k independent exponentially distributed random variables, each with rate q. i.e. $T_1 \sim \exp(k\sigma)$.
- And hence, $J_{T_1} = kT_1 \sim \exp(\sigma)$.

Continuous-time Galton–Watson processes and compound Poisson

- Apply Markov property at time T_1, when the number of individuals moves from k to $k+i$ with probability π_i, and use this same reasoning again until the second branching time, and so on....
- The time change $Z(\varphi(t))$ has the effect of spacing out branching events with independent and identical exponentially distributed random times.
- Said another way: $\{L(t) : t \geq 0\}$ is a compound Poisson process with arrival rate q and jump distribution $F(\mathrm{d}x) = \sum_{i=-1}^{\infty} \pi \delta_i(\mathrm{d}x)$.

Continuous-time Galton–Watson processes and compound Poisson

- The converse is also true: Suppose that L_t is a compound Poisson process with arrival rate q and jump distribution $F(\mathrm{d}x) = \sum_{i=-1}^{\infty} \pi\delta_i(\mathrm{d}x)$. Let

$$K_t = \int_0^t \frac{1}{L(s)}\mathrm{d}s, \qquad t \geq 0,$$

set

$$\theta(t) = \inf\{u > 0 : K_u > t\}$$

and define

$$Z(t) = L(\theta(t) \wedge \tau_0), \qquad t \geq 0,$$

where

$$\tau_0 = \inf\{t > 0 : L(t) = 0\}.$$

- Then $\{Z(t) : t \geq 0\}$ is a continuous-time Galton–Watson process.

Lamperti transform

- The same time change using the additive functional

$$\int_0^t X_s \mathrm{d}s, \qquad t \geq 0$$

  makes $X(\varphi(t))$, $t \geq 0$ a Lévy process with no negative jumps and with Laplace exponent ψ.

- Similarly, given a Lévy process $\{L(t) : t \geq 0\}$ with no negative jumps and Laplace exponent ψ, the same transform as before using the additive functional

$$\int_0^t \frac{1}{L(s)} \mathrm{d}s, \qquad t \geq 0$$

  makes $L(\theta(t) \wedge \tau_0)$, $t \geq 0$, a CSBP with branching mechanism ψ, where

$$\tau_0 = \inf\{t > 0 : L(t) = 0\}.$$

CSBP as solution SDEs

- Represent the Lévy processes with Laplace exponent ψ

$$L(t) = -at + \sigma B_t + \int_{[0,t]} \int_{|x| \geq 1} xN(\mathrm{d}s, \mathrm{d}x) + \int_{[0,t]} \int_{|x|<1} x\tilde{N}(\mathrm{d}s, \mathrm{d}x).$$

- There is a standard Brownian motion B^X, and an independent Poisson measure N^X on $[0, \infty) \times (0, \infty) \times (0, \infty]$ with intensity measure $\mathrm{d}s\mathrm{d}v\Lambda(\mathrm{d}r)$ such that

$$X_t = x + a\int_0^t X_s \mathrm{d}s + \sigma \int_0^t \sqrt{X_s}\mathrm{d}B_s^X + \int_0^t \int_0^{X_{s-}} \int_1^\infty rN^X(\mathrm{d}s, \mathrm{d}v, \mathrm{d}r) + \int_0^t \int_0^{X_{s-}} \int_0^1 r\widetilde{N}^X(\mathrm{d}s, \mathrm{d}v, \mathrm{d}r),$$

where $\widetilde{N}^X$ is the compensated Poisson measure associated with N^X.

Infinite divisibility and excursions

- The factorisation of $-\log \mathbb{E}_x[\mathrm{e}^{-\lambda X_t}]$ in to $u_t(\lambda)$ and x is a consequence of 'infinite divisibility': for $x > 0$ and any $n \in \mathbb{N}$

$$\mathbb{P}_x = \mathbb{P}_{x/n} \otimes \cdots \otimes \mathbb{P}_{x/n}$$

- It can be show that $(\mathbb{P}_x, x \geq 0)$ generates a measure $\mathbb{N}$ on

$$D_0(\mathbb{R}_+, \mathbb{R}_+) := \{\omega \in D(\mathbb{R}_+, \mathbb{R}_+) : \omega_0 = 0\}$$

such that

$$\mathbb{E}_x[\mathrm{e}^{-\lambda X_t}] = \exp\left\{\int_0^x \int_{D_0(\mathbb{R}_+,\mathbb{R}_+)} (1 - \mathrm{e}^{-\lambda\omega_t})\mathrm{d}s\mathrm{d}\mathbb{N}(\omega)\right\} = \mathrm{e}^{-u_t(\lambda)x}$$

so that

$$\mathbb{N}(1 - e^{-\lambda\omega_t}) = \int_{D_0(\mathbb{R}_+,\mathbb{R}_+)} (1 - \mathrm{e}^{-\lambda\omega_t})\mathrm{d}\mathbb{N}(\omega) = u_t(\lambda).$$

- Think Campbell formula!! See board.

CSBP with immigration

- Define a Markov process $X^* = \{X_t^* : t \geq 0\}$ on $D(\mathbb{R}_+, \mathbb{R}_+)$, with probabilities $\{\mathbf{P}_x : x \geq 0\}$, branching mechanism ψ and immigration mechanism ϕ such that:
- For all $x, t > 0$ and $\theta \geq 0$,

$$\mathbf{E}_x(\mathrm{e}^{-\lambda X_t^*}) = \exp\{-x u_t(\lambda) - \int_0^t \phi(u_{t-s}(\lambda))\mathrm{d}s\}$$

  where $u_t(\lambda)$ as before and ϕ is the Laplace exponent of any subordinator.
- Specifically, for $\theta \geq 0$,

$$\phi(\theta) = \delta\theta + \int_{(0,\infty)} (1 - \mathrm{e}^{-\theta x})\Upsilon(\mathrm{d}x),$$

  where Υ is a measure concentrated on $(0, \infty)$ satisfying $\int_{(0,\infty)}(1 \wedge x)\Upsilon(\mathrm{d}x) < \infty$.

CSBP with immigration

- Suppose that N^* is a Poisson point process with intensity

$$\left(\delta \mathrm{d}\mathbb{N}(\omega) + \int_{(0,\infty)} \Upsilon(\mathrm{d}x)\mathrm{d}\mathbb{P}_x(\omega)\right) \mathrm{d}s$$

  then we can identify the process

$$X_t^* = X_t + \int_{[0,t]} \int_{D_0(\mathbb{R}_+,\mathbb{R}_+)} \omega_{t-s}\, N^*(\mathrm{d}s, \mathrm{d}\omega), \;\; t \geq 0,$$

  where X is a CSBP issued from $X_0 = x$.

- Another way of seeing this: If $S_t = \delta t + \sum_{u \leq t} \Delta S_u$ is the subordinator with exponent ϕ, then

$$X_t^* = X_t + \sum_{u \leq t} \omega_{t-u}^{(u,\Delta S_u)} + \sum_{u \leq t} \omega_{t-u}^{(u,0)}, \qquad t \geq 0,$$

  where $\omega^{(u,\Delta S_u)}$ and $\omega^{(u,0)}$ are the points of the point process N^*, starting with positive and zero mass respectively.

Stationary subcritical processes with immigration

Theorem (M. Pinsky)

Take ψ, ϕ *and* X^* *as before (*ψ *conservative). Suppose that* $\psi'(0+) \geq 0$*. Then,* X^* *converges in distribution if and only if*

$$-\int_{0+} \frac{\phi(r)}{\psi(r)} \mathrm{d}r < \infty,$$

Spine

Theorem (Lambert)

Suppose that $X = \{X_t : t \geq 0\}$ is a conservative continuous-state branching process with branching mechanism ψ satisfying $\int^\infty \frac{1}{\psi(u)} \mathrm{d}u < \infty$. For each event $A \in \sigma(X_s : s \leq t)$ and $x > 0$,

$$P_x^\uparrow(A) := \lim_{s\uparrow\infty} \mathbb{P}_x(A | \zeta_0 > t + s)$$

is well defined as a probability measure and satisfies

$$P_x^\uparrow(A) = \mathbb{E}_x(\mathbf{1}_A \mathrm{e}^{\psi'(0+)t} \frac{X_t}{x}).$$

In particular, $\mathbb{P}_x^\uparrow(\zeta_0 < \infty) = 0$ and $\{\mathrm{e}^{\psi'(0+)t} X_t : t \geq 0\}$ is a P_x-martingale.

Spine

Lemma (Lambert)

Fix $x > 0$. Suppose that $(X, \mathbb{P}_x)$ is a conservative continuous-state branching process with branching mechanism ψ satisfying $\int^\infty \frac{1}{\psi(u)} \mathrm{d}u < \infty$. Then $(X, P_x^\uparrow)$ has the same law as a continuous-state branching process with branching mechanism ψ and immigration mechanism ϕ, where for $\theta \geq 0$,

$$\phi(\theta) = \psi'(\theta) - \psi'(0+).$$

Note that ϕ has Lévy measure $\Upsilon(\mathrm{d}x) = x\Pi(\mathrm{d}x)$.

Theorem (Fittipaldi and Fontbona)

Under $P^{\uparrow}$, the process Z is the unique strong solution of the following stochastic differential equation:

$$Z_t = x + a\int_0^t Z_s \mathrm{d}s + \sigma\int_0^t \sqrt{Z_s}\mathrm{d}B_s^{\uparrow} + \int_0^t\int_0^{Z_s-}\int_1^{\infty} rN^{\uparrow}(\mathrm{d}s,\mathrm{d}v,\mathrm{d}r) + \int_0^t\int_0^{Z_s-}\int_0^1 r\tilde{N}^{\uparrow}(\mathrm{d}s,\mathrm{d}v,\mathrm{d}r) + \int_0^t\int_0^{\infty} rN^{\star}(\mathrm{d}s,\mathrm{d}r) + \sigma^2 t,$$

where $\{B_t^{\uparrow} : t \geq 0\}$ is a Brownian motion, $N^{\uparrow}$ and $N^{\star}$ are Poisson measures on $[0,\infty)\times(0,\infty)^2$ and $[0,\infty)\times(0,\infty)$ with intensity measures $\mathrm{d}s\times\mathrm{d}v\times\Pi(\mathrm{d}r)$ and $\mathrm{d}s\times r\Pi(\mathrm{d}r)$, respectively, and these objects are mutually independent (as usual, $\tilde{N}^{\uparrow}$ represents the compensated measure associated with $N^{\uparrow}$).

Skeleton (or backbone)

A conservative supercritical CSBP X with branching mechanism ψ under $\mathbb{P}_x$, $x > 0$ can be identified as equal in law to the following construction.

- Let λ^* be the solution to the equation $\psi(\lambda^*) = 0$. Let N be an independent Po$(\lambda^* x)$ r.v.
- Initiate N independent Galton-Watson processes with branching mechanism

$$F(s) = q \sum_{n \geq 0} p_n (s^n - s) = \frac{1}{\lambda^*} \psi(\lambda^*(1-s)),\ s \in (0,1),$$

where the individual components of F are given by $q = \psi'(\lambda^*)$, $p_0 = p_1 = 0$ and for $n \geq 2$,

$$p_n = \frac{1}{\lambda^* \psi'(\lambda^*)} \left\{ \sigma(\lambda^*)^2 \mathbf{1}_{\{n=2\}} + (\lambda^*)^n \int_{(0,\infty)} \frac{x^n}{n!} e^{-\lambda^* x} \Pi(\mathrm{d}x) \right\}.$$

Skeleton (or backbone)

- Along the edges of the space-time graph of the N Galton-Watson trees, immigrate CSBPs $\omega.$ at rate

$$2\sigma\mathbb{N}^*(\mathrm{d}\omega) + \int_{(0,\infty)} y e^{-\lambda^* y}\Pi(\mathrm{d}y)\mathbb{P}^*_y(\mathrm{d}\omega)$$

  where $\mathbb{P}^*_x$, $x \geq 0$ is the family of laws associated to the CSBP with branching mechanism $\psi^*(\lambda) = \psi(\lambda + \lambda^*)$ (corresponding to X conditioned to die out - so, in the appropriate sense, $\mathbb{P}^*_y = \mathbb{P}^{\downarrow}_y$) and $\mathbb{N}^*$ is the associated excursion measure.

- Moreover, at any branch point, given that $n \geq 2$ offspring are produced, then an additional and indpendent $\mathbb{P}^*_y$ branching process is immigrated with probability

$$\eta_n(\mathrm{d}y) = \frac{1}{p_n \lambda^* \psi'(\lambda^*)}\left\{\beta(\lambda^*)^2\delta_0(\mathrm{d}y)\mathbf{1}_{\{n=2\}} + (\lambda^*)^n \frac{y^n}{n!}e^{-\lambda^* y}\Pi(\mathrm{d}y)\right\}.$$

Skeleton (or backbone)

- Finally add an independent copy of $(X, \mathbb{P}^*_x)$ to the Poisson number of 'dressed' Galton-Watson trees and this is what $(X, \mathbb{P}_x)$ is equal to in law.