Convergence of weighted trees

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Ising model

Let $G = (V, E)$ be a graph. For $\beta \geq 0$, define a measure $P_{\beta, G}$ on $\{-1, 1\}^V$ by setting

$$P_{\beta, G}(\sigma) = \frac{1}{Z} \exp \left\{ \sum_{v \sim w} \beta \sigma_v \sigma_w \right\}.$$

![Diagram of a graph with vertices and edges]
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- \( \beta = 0 \): uniform pick
- \( \beta \uparrow \infty \): pick + or − with pba 1/2 and give all the vertices same spin
Big problem

- For each $n$, discretise the sphere $S^2$ to get a finite set $\mathcal{M}_n$ (for example quadrangulations)
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For \( G \in \mathcal{M}_n, \sigma \in \{-1, 1\}^V \):

\[
\mathbb{P}_\beta((m_n, \sigma_n) = (G, \sigma)) = \frac{1}{Z} \mathbb{P}_{\beta, G}(\sigma)
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Find a scaling limit of $m_n$. 
Think of $m_n$ as a metric space with graph distance $d_{gr}$.
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**Theorem (Miermont (2013), Le Gall (2013))**

When $\beta = 0$ (picking uniformly), for a large class of discretisations

$$\lim_{n \to \infty} \left( \mathbf{m}_n, \frac{1}{n^{1/4}} d_{gr} \right) = (\mathbf{m}_*, d)$$

in distribution under the Gromov-Hausdorff topology. The limiting space $(\mathbf{m}_*, d)$ is called the Brownian map.
Proposed problem

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Let $\mathcal{T}_n$ be the set of labelled trees with $n$ vertices and pick $(T_n^{(\beta)}, \sigma_n)$ by

$$\mathbb{P}_\beta((T_n^{(\beta)}, \sigma_n) = (T, \sigma)) = \frac{1}{Z} \mathbb{P}_{\beta, T_n^{(\beta)}}(\sigma)$$

(possibly with boundary conditions).

**Question**

Does there exists a space $(T^{(\beta)}, d)$ and a $\gamma > 0$ such that

$$\lim_{n \to \infty} (T_n^{(\beta)} , n^{-\gamma} d_{gr}) = (T^{(\beta)}, d)$$

in distribution?
What is known?

**Theorem (Aldous (1991))**

When $\beta = 0$ (picking uniformly),

$$\lim_{n \to \infty} \left( T_n^{(0)}, \frac{1}{n^{1/2}} d_{gr} \right) = (T, d)$$

_in distribution under the Gromov-Hausdorff topology. The limiting space $(T, d)$ is called the continuum random tree._
An approach is to take an exploration of the tree
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An approach is to take an exploration of the tree and try to incorporate the Ising model when exploring.
Thank You!